Effective elastic parameters of the two-dimensional phononic crystal

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The effective acoustic velocity along two highly symmetric directions of the first Brillouin zone is studied in a two-dimensional squarely arranged solid-solid phononic crystal. The relationship between the effective elastic velocity and the wave frequency of the two lowest bands is presented, which gives the same results as those obtained directly from the band structure. Numerical calculations show that the effective acoustic velocity of the phononic crystal decreases with frequency increasing from zero to the band edge.

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It is well known that the effective medium theory of the composite material is based on the long-wavelength limit. In this limit the dimension of scatterers goes beyond the resolution of the wave, and the studied system is considered to be homogenous, so its wave behavior can be described by effective parameters.

Several methods have been developed to calculate the effective sound velocity of the periodic composite system such as bubbly water [1–3] and rigid cylinders in air [4,5]. As was pointed out in Ref. [1], the long-wavelength limit requires that the wavelength of a propagating wave should be more longer than the distance of adjacent scatterers, which means that the resonance resulting from standing waves between adjacent scatterers can be neglected. But the fact is that most of the interesting acoustic phenomena, such as the appearance of a band gap [6,7] and the elastic wave bending effect [8], occur only if the wavelength is comparable to or even smaller than the distance between adjacent scatterers. Therefore, it is significant if the effective medium theory can be extended to a higher-frequency regime. Some efforts have been done to study the unusual conductance of photonic crystals [9,10], where an effective refractive index less than unity and a negative refractive index were defined. The main idea is that if the wave propagating process is disregarded and Snell's law is employed at boundary, then the unusual behavior of the wave in photonic crystal can be described by an effective index. Usually, the refractive index of a conventional optical medium can be obtained by the formula $n = ck/\omega$, but a further consideration is needed if this formula is extended to the photonic crystal because of the nonlinearity of its dispersion relation and the undetermined direction of its transmitting wave. In Ref. [10], Notomi clarified the features of light propagation in photonic crystals and showed that, despite the existence of diffractive features, the light propagation in strongly modulated photonic crystal becomes refraction like in the vicinity of the band gap and the effective refractive index can be well defined in these frequency regimes.

The above results on photonic crystals can also be extended to phononic crystals. But in the latter case the problem becomes more complicated since there exists more than one dispersion relationship even in a uniform medium. We note that a slab of phononic crystal is something like a surface grating. When a plane wave impacts on the surface of the slab, the transmitting and reflecting wave usually consists of a series of plane waves with different propagating directions, which are named the 0th-, ±1st-, ±2nd-, …, and ±nth-order diffractive waves. But if the transmitting wave includes only one nonzero component (the 0th, for example), then it can be regarded as a traditional refractive wave, and in this case the effective elastic parameters can be defined.

In the present work, we use a method similar to that used in Refs. [4,5] to calculate the effective velocity of a two-dimensional solid-solid phononic crystal slab. We first calculate the transmission spectra accurately by the eigenmode matching theory (EMMT) [11]. The advantage of the EMMT is that the energy flux for all orders of the diffractive waves can be simultaneously obtained, which helps us decide in which frequency regime the effective parameters can be well defined. For simplicity, we restrict our calculation to the frequency regime of the two lowest bands, which are under the frequency of the diffractive limit. The effective acoustic velocities along the ΓX and ΓM directions are then obtained from the transmission spectrum. As a comparison, we also calculate the effective velocity from the band structure of the infinite system with the same configuration by using the formula $c=\omega/k$. The results from these two methods agree surprisingly with each other, which implies that the effective acoustic velocity in the phononic slab can be roughly obtained by the band structure of the infinite system.

Now we use the EMMT to study the transmission spectrum of the two-dimensional phononic crystal schematically shown in Fig. 1, which is infinite along the z and x directions, and have n unit cells along the y direction. The elastic plane wave from the leftmost semi-infinite uniform substrate impacts normally on the surface of the slab. To use the EMMT method, each unit cell has to be cut into many slices along the y direction, and each slice is approximately considered as a periodic layer-stacked system along the x direction.

In a two-dimensional system, the xy and z modes can be decoupled [12,13] and only the xy mode in the phononic crystal can be excited when the incident wave has in-plane polarization. So we will consider the xy mode only in the following for simplicity.

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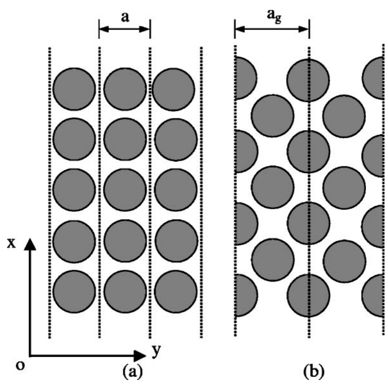

FIG. 1. Two-dimensional cross sections of the square array of parallel circular lead rods with radius r embedded in an epoxy matrix. The system is infinite in the x and z directions. The unit cell is shown by the dotted line. (a) Cross section to calculate the transmission spectrum along the ΓX direction. a is the lattice constant. (b) For the ΓM direction, $a_g=\sqrt{2}a$ is the lattice constant.

As is shown in Ref. [11], the wave solution in each slice and in both of the semi-infinite substrates can be expressed as

$$\begin{pmatrix} \mathbf{U} \\ j\mathbf{T_2} \end{pmatrix} = \begin{pmatrix} e^{j\alpha x} & 0 & 0 & 0 \\ 0 & e^{j\alpha x} & 0 & 0 \\ 0 & 0 & e^{j\alpha x} & 0 \\ 0 & 0 & 0 & e^{j\alpha x} \end{pmatrix} \begin{pmatrix} u_R & u_L \\ t_{2R} & t_{2L} \end{pmatrix} \begin{pmatrix} e^{j\beta_R y} & 0 \\ 0 & e^{j\beta_L y} \end{pmatrix} \times \begin{pmatrix} A_R \\ A_L \end{pmatrix}. \tag{1}$$

Here, $\mathbf{U}=(U_1,U_2)^t$ and $\mathbf{T_2}=(T_{21},T_{22})^t$ are the displacement and stress vectors, respectively. $e^{j\alpha x}$ is a $N\times N$ diagonal matrix and $e^{j\beta y}$ is a $2N\times 2N$ diagonal matrix. u and t_2 are the $2N\times 2N$ eigenvector matrix associated with β, and the subscripts R and L stand for right and left, respectively. A_R and A_L are the $2N\times 1$ amplitude vectors corresponding to the right- and left-forward waves. For the incoming plane wave, only the 0th Fourier component of A_R takes a nonzero value and, for the outgoing wave, the amplitude $A_L(m)=0$. For the solid-solid phononic crystal with small contrast of elastic parameters, a reliable result in the low-frequency regime can be obtained by using a smaller N.

The transmission (reflection) coefficient of Fourier component can be defined as the ratio of the average transmission (reflection) energy flux with the average incoming energy flux along the y direction,

$$T_i = \left| \frac{\mathrm{Re}[(\dot{U}_i^{out})^* T_{2i}^{out}]}{\mathrm{Re}[(\dot{U}^{in})^* \cdot T_2^{in}]} \right|, \quad R_i = \left| \frac{\mathrm{Re}[(\dot{U}_i^{ref})^* T_{2i}^{ref}]}{\mathrm{Re}[(\dot{U}^{in})^* \cdot T_2^{in}]} \right|, \tag{2}$$

where $(\dot{U}_i)^*$ denotes the conjugate of the ith component of the vector $\dot{U}$ and "Re" takes the real part of a complex value. The superscripts *out*, *in*, and *ref* indicate the outgoing, incoming, and reflecting waves, respectively. The same as the definition in the surface grating [14], T_i is called the diffractive ratio of ith order. The energy conservative condition $\Sigma_i T_i+\Sigma_i R_i=1$ must be satisfied in the calculation procedure.

If T_i has only one nonzero component, such as T_0, T_1, or T_{-1}, the behavior of the wave through the phononic crystal slab can be considered as a refractive phenomenon, and the effective elastic parameters can then be defined.

To show how to obtain the effective velocity of the phononic crystal, we first consider a system formed by a uniform medium slab with acoustic impedance Z_2 sandwiched by two semi-infinite uniform substrates with acoustic impedance Z_1. The energy transmission coefficient of the system for the normal incidence can be written as [15]

$$T = \frac{1}{\cos\left(\frac{\omega}{c}L\right)^2 + \left(\frac{Z_1^2+Z_2^2}{2Z_1Z_2}\right)^2 \sin\left(\frac{\omega}{c}L\right)^2}, \tag{3}$$

where ω is the angular frequency of the incident wave, c is the acoustic velocity, and L is the thickness of the sandwiched slab, respectively.

From Eq. (3), we see that T is a periodic function of ω/c for a given L, which has the maximum value 1 if ω_i satisfies

$$\frac{\omega_i}{c}L = n\pi \quad (n=0,\pm 1,\dots). \tag{4}$$

The above two equations suggest that if the transmission spectrum can be calculated through a different way and the frequencies (ω_i) with $T=1$ can be selected out, then the acoustic velocity c corresponding to ω_i will be obtained by

$$c(\omega_i) = \omega_i L/n\pi \quad (n=0,\pm 1,\dots) \tag{5}$$

for a given thickness L.

The minimum values of T are determined by the mismatch of Z_1 and Z_2: when $Z_1/Z_2\rightarrow 0$ or $Z_1/Z_2\rightarrow\infty$, $T\rightarrow 0$. For a given Z_1/Z_2, T takes the minimum value when ω_i satisfies

$$\frac{\omega_i}{c}L = (n+1/2)\pi \quad (n=0,\pm 1,\dots). \tag{6}$$

If a phononic crystal slab can be considered as an effective uniform medium with effective acoustic impedance Z_{eff} and effective velocity c_{eff}, then its transmission coefficient can also be expressed by Eq. (3) with Z_2 and c replaced by Z_{eff} and c_{eff}. We cannot expect Z_{eff} and c_{eff} to be independent of ω, but a simple calculation shows that Eq. (4) can still be used to determine the corresponding angular frequencies (ω_i) for $T=1$. On the other hand, Eq. (6) becomes invalid because of the frequency dependance of c_{eff} and Z_{eff}.

From Eq. (5) we see that to obtain the effective velocity c_{eff}, the integer n must be determined beside the angular frequency ω_i. The corresponding phase shifts between the nearest peaks of T should be π, so if we can determine one of these integers n (like a seed) and the trend of phase shifting with ω increasing or decreasing, then the rest of n can be determined. Because only the two lowest bands are considered in the present paper, the transmission spectra start from $\omega=0$ and the corresponding n takes zero, so the following n should be 1,2,…, when ω increases because of the positive c_{eff}.

Using the method presented above, we investigate a two-

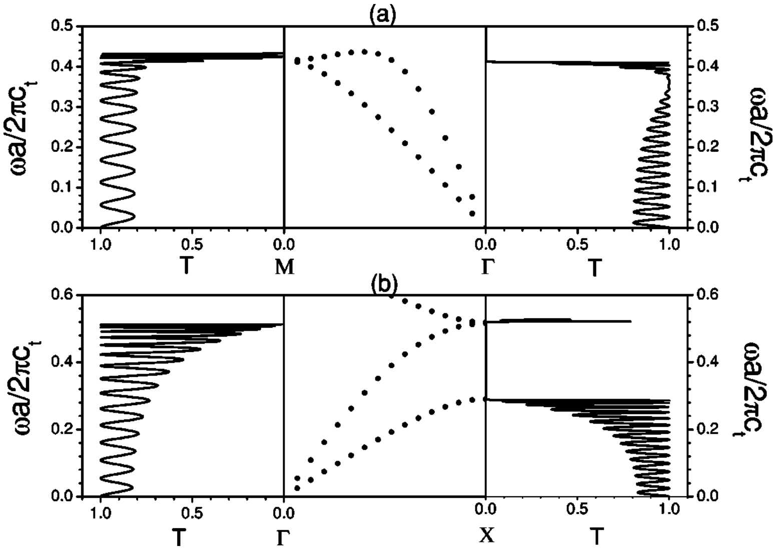

FIG. 2. The transmission spectra with filling fraction $f=0.126$ $(r/a=0.2)$. The frequency is scaled as $\omega a/2\pi c_t$ (a) for wave propagating along the ΓM and (b) for the ΓX direction. The left (right) panels correspond to longitudinal (transversal) normally incident wave. The middle panels are the band structures of the infinite system.

dimensional composite constructed by the lead circular rods squarely embedded in the epoxy matrix as shown in Fig. 1. The unit cells are divided into 100 or more slices. Here 21 plane waves are used to expand the wave solution along the x direction in each slice. For the two lowest bands, only the zeroth-order T_0 and R_0 exist, which means no diffractive effect occurs.

A typical transmission spectrum for the system with $r/a=0.2$ is presented in Fig. 2. The band structure corresponding to the middle panels is obtained by the plane-wave expansion method. The thickness of the slabs are 15 unit cells $(15a)$ along the ΓX and 10 unit cells $(10\sqrt{2}a)$ along the ΓM direction, respectively. The first feature extracted from the figure is that the transmission spectra coincide well with the band structures and the lowest (second lowest) branch in the band can only be excited by the transversally (longitudinally) polarized incoming wave, which means that in the studied frequency regime phononic crystal behaves like the conventional elastic medium. The second feature is that except in the region around $\omega a/2\pi c_t=0.42$ on the left panel of Fig. 2(a), the transmission spectra fluctuate like a periodic function and the period decreases as the angular frequency increases, which indicates that the effective velocity is dependent on the frequency of propagating wave. The spectrum for the longitudinally polarized wave along ΓM direction shows an irregular fluctuation as its frequency approaches the band edge (around $\omega a/2\pi c_t=0.42$). This irregular fluctuation results from the unmonotonous feature of the corresponding branch in the band structure, a discussion for which will be presented in the following.

The effective velocities corresponding to the two lowest bands along the ΓX direction are shown in Fig. 3 (solid line) and the values calculated by $c_{eff}=\omega/k$ are also presented (circles) as a comparison. We can see that the two results coincide with each other very well except for the system with $r/a=0.49$, the difference of which may be caused by the convergence of the plane-wave method. This remarkable coincidence shows that for the studied system the effective velocities of these two branches can be directly calculated from the band curves, even though it has a large deviation from the straight line. Figure 3 also shows that both longitudinal and transversal effective velocities decrease with frequency increasing and the decrease becomes faster as the corresponding wave vector gets closer to the edge of the first Brillonin zone (BZ). From the figure we also see that in the small frequency regime, the curves for the system with low (such as $r/a=0.1$) or high (such as $r/a=0.49$) filling fraction

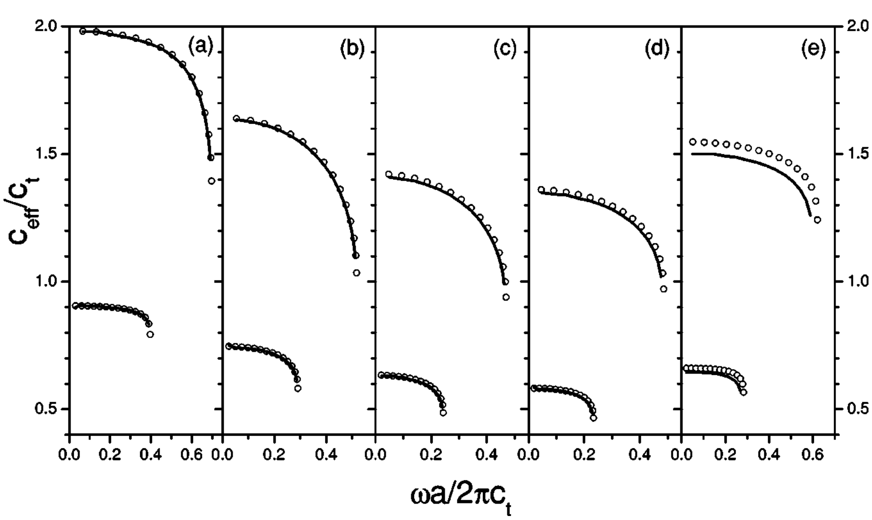

FIG. 3. Effective velocity of the two lowest bands along the ΓX direction. (a)–(e) correspond to the system with $r/a=0.1$, 0.2, 0.3, 0.4, and 0.49, respectively. The upper (lower) curve is for the the longitudinal (transversal) wave, solid and circle curves, respectively, representing the effective velocities obtained from transmission spectra and band structures.

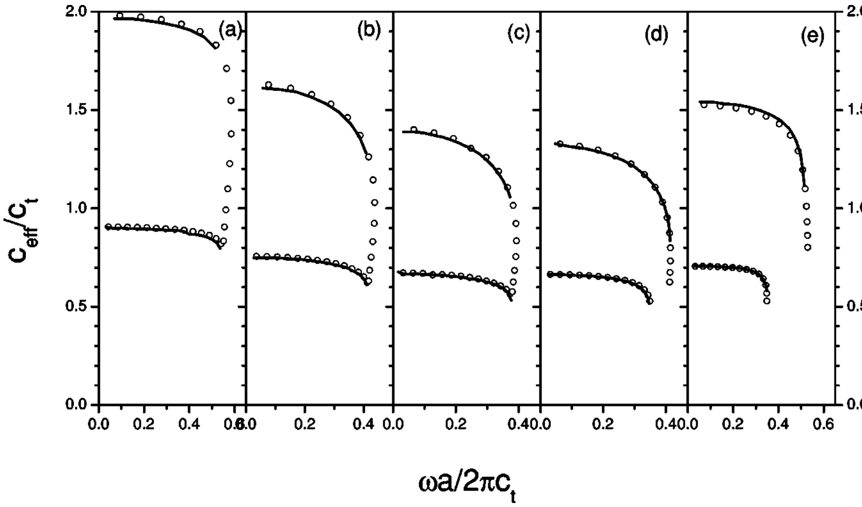

FIG. 4. Same as Fig. 3 for the ΓM direction.

are more flat than the ones for the system with moderate filling fractions, which is even true for the transversal wave.

The effective velocities corresponding to the two lowest bands along ΓM direction are shown in Fig. 4, which gives a similar conclusion as that of Fig. 3. Particularly, we would like to point out that for the longitudinal wave, if the filling fraction is smaller than a certain value, the method presented loses its validity in the frequency regime close to the band edge. Therefore, we only show the effective velocities in lower-frequency segments in Figs. 4(a)–4(c). We note that the branch of the band in the middle panel of Fig. 2(a) does not increase monotonically when the wave vector increases along the ΓM line. A peak appears around the middle of ΓM, which leads to two ω/k values for one ω [as shown by circles in Figs. 4(a)–4(c)]. This means that there exist two effective velocities when the frequency closes to the peak. This phenomenon will disappear when the value of r/a becomes large [e.g., $r/a=0.4$ and 0.49 shown in Figs. 4(d) and 4(e), respectively]. In the case of bivelocity, Eq. (3) and then Eq. (4) cannot be used anymore to describe the wave behavior of the phononic crystal slab. Systems with bivelocity are also discussed by Liu *et al.* [16], but what they reported is obviously not the same as presented here. A further discussion for this bivelocity phenomenon would appear elsewhere.

Another feature drawn from Figs. 3 and 4 is the anisotropic property of the acoustic velocity of the system studied. As was pointed out in Ref. [5], a phononic crystal with square-lattice structure should have an elastic anisotropic property. Our results show that the anisotropy of the studied system depends on the filling fraction and the system with moderate filling fraction (e.g., $r/a=0.3$) would have the strongest anisotropic elastic property.

Finally we would like to point out that the effective velocity of the phononic crystal obtained by the method in Refs. [4,5] should correspond to the flat part of the curves (near $\omega a/2\pi c_t=0$) shown in Figs. 3 and 4, where the long-wavelength limit is valid.

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